Engineering Notes

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An Algorithm for an Approximation of the Minimal Controller Problem

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Introduction

IN this Note a systematic approach is sought to make a compromise between control system performance and the order of the controller design model. The modeling problem and the control problem are not independent, and a merger of the two problems is approximated in a two-step design: 1) a model reduction problem is solved using performance sensitivity criteria[1]; 2) a model error compensating state estimator is constructed, using orthogonal functions to fit the unknown model error vector in real time.[2] The resulting controller design may take on two forms: a linear dynamical feedback controller, or an adaptive "model learning" controller. An implementation procedure is described.

To evaluate candidate control policies for elastic bodies by computer simulation a large number of elastic modes are usually retained in the "evaluation model," S_I,

$$S_I \begin{cases} \dot{x}_I = A_I x_I + B_I u + \Gamma_I w + f(x_I,u,t), & x_I \epsilon R^{n_I} \\ y_I = C_I x_I, & y_I \epsilon R^k \\ z_I = M_I x_I + v, \quad w \epsilon R^{\omega}, \quad u \epsilon R^m, & z_I \epsilon R^{\ell} \end{cases} \tag{1}$$

where y_I represents the k outputs to be regulated to zero, and z_I represents the ℓ measurements actually available. The disturbances w and v are assumed to be zero mean white noise processes. Any other disturbances are presumed modeled by a set of differential equations which have already been augmented to the system model to form the composite model (1). The control policy will be based upon a much simpler linear "design model," S_j,

$$S_j \begin{cases} \dot{x}_j = A_j x_j + B_j u + \Gamma_j w, & x_j \epsilon R n_j, \quad n_j < n_I \\ y_j = C_j x_j & y_j \epsilon R^k \\ z_j = M_j x_j + v & z_j \epsilon R^{\ell} \end{cases} \tag{2}$$

for practical and economical synthesis of the linear dynamical feedback controller S_c,

$$S_c \begin{cases} \dot{\hat{x}}_j = A_j \hat{x}_j + B_j u + \hat{G}(z_I - M_j \hat{x}_j), & \hat{x}_j \epsilon R^n j \quad \text{(3a)} \\ u = -G\hat{x}_j & \text{(3b)} \end{cases}$$

The dynamical feedback controller (3) is of order n_j and is composed of a full-order state estimator [Eq. (3a)] and a linear control law [Eq. (3b)]. The state estimation gain $\hat{G}$ and the control gain G are usually determined so that the composite closed-loop "design model" described by Eqs. (2) and (3) is at least stable (e.g., by using pole assignment methods to specify G and $\hat{G}$) or, perhaps optimal (e.g., by using the linear regulator and estimation theory of optimal control to specify G and $\hat{G}$). The inevitable errors in the design model (truncated modes, parameter errors, neglected disturbances, neglected nonlinearities) prevent the performance claims of modern control theory from being realized for the evaluation model S_I (which utilizes the controller S_c, which was based upon model S_j). The closed-loop evaluation S_{Ic} is described by Eqs. (1) and (3), or, equivalently, by

$$S_{Ic} \begin{cases} \dot{x}_c = A_c x_c + \Gamma_c w_c + \Sigma_c f \\ y_I = C_c x_c \\ u = -G_c x_c \end{cases} \tag{4}$$

where

$$A_c = \begin{bmatrix} A_I & -B_j G \\ \hat{G}M_I & A_j - \hat{G}M_j - B_j G \end{bmatrix}, \quad x_c = \begin{pmatrix} x_I \\ \hat{x}_j \end{pmatrix}$$

$$\Sigma_c = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad w_c = \begin{pmatrix} w \\ v \end{pmatrix}, \quad \Gamma_c = \begin{bmatrix} \Gamma_I & 0 \\ 0 & \hat{G} \end{bmatrix}$$

$$G_c \begin{bmatrix} 0 & I \end{bmatrix}, \qquad C_c = \begin{bmatrix} C_I & 0 \end{bmatrix}$$

It is not generally clear how to find the "best" design model S_j which allows the physical system (which is assumed to be reliably modeled by S_I) to meet the control objectives.

The performance of system (4) is evaluated by the performance measure

$$V_{Ij} = \frac{I}{2} \int_0^T (\|y_I\|_Q + \|u\|_R) \mathrm{d}t, \quad \|u\|_R = u^T R u \tag{5}$$

The performance measure V_{ij} is quadratic in the outputs y_I and the controls u of model S_I, given that the controller design is based upon model S_j. The scalar V_{Ij} is influenced by both the control policy and the model S_j upon which the control policy is based, $u = -G(S_j) \cdot \hat{x}_j(S_j, z_I)$. However, by fixing the algorithms which specify the control gains G and the estimator gains $\hat{G}$ in terms of parameters of the reduced model S_j, the scalar V_{Ij} becomes a measure of quality of the model S_j. For our purposes, the control and estimation policies are fixed by linear regulator and estimation theory (to minimize V_{jj} as $T \to \infty$),

$$G = R^{-I} B_j^T K, \quad 0 = -KA_j - A_j^T K + KB_j R^{-I} B_j^T K - C_j^T Q C_j \tag{6a}$$

$$\hat{G}^T = \hat{R}^{-I} M_j \hat{K}, \quad 0 = -\bar{A}_j \hat{K} - \hat{K}\bar{A}_j^T + \hat{K} M_j^T \hat{R}^{-I} M_j \hat{K} - \Gamma_j \hat{Q} \Gamma_j^T$$

$$\bar{A}_j = A_j + I\, 1/\tau \tag{6b}$$

under the presumptions that 1) the uncontrollable modes of the pair (A_j, B_j) are stable, 2) the pair (A_j, C_j) is observable, 3) the unobservable modes of (A_j, M_j) are stable, 4) the pair

Received July 7, 1977; revision received Aug. 25, 1977.

Index category: Guidance and Control.

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(A_j, Γ_j) is controllable,† and 5) the matrices, R, Q, $\hat{R}$, $\hat{Q}$, are all positive definite. These conditions insure that the optimal control gain G stabilizes the matrix $[A_j - B_jG]$, and the state estimator gain $\hat{G}$ assures that $-Re\lambda_i[A_j - \hat{G}M_j] > 1/\tau$ for all $i = 1, 2, \ldots n_j$ for any selected τ. The problem of selecting model S_j remains. The control algorithm is now fixed, and the scalar V_{lj} now becomes a "model quality index." Since S_l is presumed more reliable than any reduced model S_j, the minimum value of the model quality index V_{lj} is V_{ll}. The minimal controller problem (MCP) may be stated: "Find the model S_j of lowest order n_j such that $V_{lj} \leq \bar{V}$, where $\bar{V}$ is a specified positive number." If controller synthesis cost is much more important than performance, one can specify an aribitrarily large number for $\bar{V} \to \infty$. If performance is much more important than synthesis cost, then specifying $\bar{V} = V_{ll}$ leads to the standard optimal design for S_l which requires a controller of order n_l. One is usually interested in performance and synthesis costs in between these two extremes. This Note will not attempt to solve the MCP directly, but will seek an approximation which is constructed in two separate phases: phase I consists of a model reduction motivated by the desire to decrease n_j, and phase II consists of a model error compensation motivated by a desire to decrease V_{lj} by designing a controller which partially compensates for the errors inherent in the reduced model S_j. In phase I, a linearized model S_l is reduced to the model which is labeled S_3 of order $n_3 < n_l$, where S_3 has the form of Eq. (2) with $j \triangleq 3$. Let the "model error vectors" $e_{xy}(t)$ and $e_z(t)$, which are unknown a priori, be defined such that $y_3(t) = y_l(t)$, and $z_3(t) = z_l(t)$ if e_{xy} is added to the state equation, and $e_z(t)$ is added to the measurement equation of S_3. Reference 1 presents the minimum performance sensitivity approach to model reduction to be used in phase I.

In phase II certain "synthetic modes" are added to the truncated model S_3 to compensate approximately for the unknown modeling errors (truncated modes, parameter errors, and neglected disturbances) associated with model S_3. The synthetic modes which accomplish this are associated with a "model error system" S_e which generates orthogonal functions to approximate $e_{xy}(t)$ and $e_z(t)$. The model error system S_e

$$S_e \begin{cases} \dot{s} = Ds \\ e = Ps \qquad \bar{e}^T = (e_{xy}^T, e_z^T), \quad P^T = [P_x^T, P_z^T] \end{cases} \tag{7a}$$

(where e is an approximation of $\bar{e}$) is augmented to the truncated model S_3 to form the controller design model S_2:

$$S_2 \begin{cases} \dot{x}_2 = A_2x_2 + B_2u + \Gamma_2 w & x_2 \epsilon R^{n_2} \\ y_2 = C_2x_2 & n_l > n_2 \geq n_3 \\ z_2 = M_2x_2 + v \end{cases} \tag{7b}$$

where

$$A_2 = \begin{bmatrix} A_3 & P_x \\ 0 & D \end{bmatrix}, B_2 = \begin{bmatrix} B_3 \\ 0 \end{bmatrix}, \begin{bmatrix} C_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} C_3 & 0 \\ M_3 & P_z \end{bmatrix}$$

$$\Gamma_2 = \begin{bmatrix} \Gamma_3 \\ \Gamma_e \end{bmatrix}$$

†Controllability of the pair $(A,B) \to$ rank $[B, AB, \ldots A^{n-1}B] = n$. Observability of the pair $(A,C) \to$ rank $[C^T, A^TC^T, \ldots A^{(n-1)T}C^T] = n$. In the case of diagonal Jordan forms, the unobservable modes of (A,M) are stable if $Re\lambda_i[A] < 0$ for any i for which $M\xi^i = 0$, where ξ^i is the eigenvector associated with the eigenvalue λ_i. The uncontrollable modes of (A,B) are stable if $Re\lambda_i[A] < 0$ for any i for which $r^{i^T}B = 0$, where r^i is the reciprocal base vector associated with λ_i.

The methods of Ref. 3 will be used to determine P_x, P_z, D, and Γ_e so that the state estimator converges [that is, $\hat{x}_3(t) \to x_3(t)$] in the presence of "almost all" modeling errors in the truncated model S_3. A step-by-step implementation procedure suitable for programming on a computer is presented.

Phase I – Model Reduction Problem

The following method of model reduction is developed in Ref. 1 and requires truncation of Jordan modal coordinates. First, the complex Jordan form of S_l is reduced and converted to real form yielding the result

$$A_3 = H^{-1}\bar{W}A_l\bar{M}H, \quad B_3 = H^{-1}\bar{W}B_l, \quad \Gamma_3 = H^{-1}\bar{W}\Gamma_l \tag{8a}$$

$$C_3 = C_l\bar{M}H, \quad M_3 = M_l\bar{M}H \tag{8b}$$

where H is a block diagonal matrix having blocks H_α defined by $H_\alpha = 1$ if λ_α is real, and

$$H_\alpha = \tfrac{1}{2} \begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix}$$

if λ_α is complex (distinct), where λ_α is the αth eigenvalue of the reduced Jordan form $\bar{W}A_l\bar{M}$. The vectors ξ^i, r^i appearing in

$$\bar{M} = [\ldots \xi^i \ldots]_{n_l x n_3} \qquad \bar{W}^T = [\ldots r^i \ldots]_{n_l x n_3} \tag{8c}$$

are the ith eigenvectors and reciprocal base vectors, respectively, of A_l, where the index i belongs to the set of retained modes (which modes are defined as having the n_3 largest truncation indices μ_i). The truncation index μ_i is defined by the first-order sensitivity of the optimal model quality index V_{ll},

$$\mu_i = \left| \frac{\partial V_{ll}}{\partial X_{0_{ii}}} \right|, \qquad X_{0_{ii}} = E[x_i^2(0)] \tag{9a}$$

$$x_l = Mx \qquad M = [\xi^1, \xi^2, \ldots, \xi^{n_l}] \tag{9b}$$

If the first term in V_{lj} (involving only the norm of the output, $\|y_l\|_Q$) dominates the model quality index V_{lj}‡, or more formally, if $\|R\|/\|Q\| \to 0$, then it has been shown that,[1]

$$\mu_i = \left| \frac{\partial V_{ll}}{\partial X0_{ii}} \right| = \beta_i \|\mu_0^i\|_Q \tag{10}$$

where $\beta_i = -1/Re\lambda_i$ and the scalar

$$\|\mu_0^i\|_Q = \|C_l\xi^i\|_Q = \xi^{i^T} C_l^T Q C_l \xi^i \tag{11}$$

is a measure of the degree of observability of mode i in the output vector y_l [$\|\mu_0^i\| = 0$ indicates unobservability of mode i in $y_l(t)$]. In this case $(R \to 0)$, the reduced mode is composed of those modes which are "most observable" in $y_l(t)$. Through the procedure of this section, the modeling problem – find $S_3(A_3, B_3, \Gamma_3, M_3)$ – and the control problem – find $u(S_3, z_l)$ – have been combined. The MCP approximation which has been solved involves the control problem through the use of C_l, R, and Q. This interdependence between the modeling problem and the control problem distinguishes this approach from other model reduction schemes.[4-8]

‡For instance, in analysis problems, where no control design is required and only a reduced model for analytical predictions of $y_l(t)$ is of interest, the model quality index V_{lj} takes on the form of Eq. (5) with $R = 0$, and the model reduction methods of this section still apply.

Phase II – Model Error Compensation

The technique of model error compensation developed in Refs. 2 and 3 seeks to approximate the model error vectors $e_{xy}(t)$ and $e_z(t)$ in Eq. (5) with orthogonal functions and to use this approximation to improve the state estimate $\hat{x}_3$ and the control $u(t)$. These concepts have led to the "orthogonal filter," and the special case of model error approximation with Chebyshev polynomials is called the "Chebyshev filter."[3] The full-order Chebyshev filter has the form of Eq. (3), where the design model S_j is taken as S_2 given by Eq. (6b) with

$$D=\frac{2}{\tau}\begin{bmatrix} 0 & 0 & 0 & \\ 1 & 0 & 0 & \\ 0 & 4 & 0 & \\ & & & \ddots \end{bmatrix}_{d\times d} \tag{12}$$

τ represents an "observation window" over which the model error approximating functions [eigenfunctions of Eq. (6a)] are orthogonal. If D is given by Eq. (12), then eigenfunctions of Eq. (6a) are Chebyshev polynomials of the first kind and the existence of a convergent Chebyshev filter [state estimator, Eq. (3), using Eqs. (6b) and (12)] in the presence of unknown modeling errors in S_3 is guaranteed $[\hat{x}_3(t)\to x_3(t)]$ in Ref. 3 under certain conditions. Practical considerations have led to these further constraints: 1) τ^{-1} lies within the spectral radius of A_1 ($|\lambda[A_1]|_{\min}<\tau^{-1}<|\lambda[A_1]|_{\max}$), 2) $\hat{G}$ is chosen so that the dominant eigenvalue of $[A_2-\hat{G}M_2]$ satisfies $\tau^{-1}\ll|\lambda[A_2-GM_2]|_{\min}$, 3) the matrix P is chosen so that the pair (P,D) is observable. (Other choices for $P^T\triangleq[[P_x^T P_z^T]$ are given in Ref. 3, including an adaptive version of P.)

Implementation Procedure

Implementation of the ideas presented may be accomplished in the following steps.

1) Select the parameters of the evaluation model S_1, the weighting factors Q, R for the control problem, and the performance requirement $\bar{V}$.

2) Perform a model reduction from S_1 to S_3 setting $n_3=1$ or 2, depending upon whether the model reduction criterion indicates that the most significant mode is associated with a real or complex eigenvalue, respectively.

3) Augment a model error system S_e of order d to S_3 to form S_2. (On the first iteration there is no model error system and d is set to $d=0$.)

4) Compute the control and estimator gains G and $\hat{G}$ using Eq. (5b) or using pole placement techniques, being careful to observe the constraints on $\hat{G}$ mentioned earlier. Choose τ and $\hat{G}$ such that

$$\tau^{-1}\triangleq\tfrac{1}{2}\{|\lambda[A_1]|_{\max}+|\lambda[A_1]|_{\min}\}$$
$$\leq 0.1\,|\lambda[A_2-\hat{G}M_2]|_{\min}$$

5) Evaluate V_{12}. If $V_{12}\leq\bar{V}$, then stop. If $V_{12}>\bar{V}$, set $d=d+1$ and return to step 3. Continue until $V_{12}\leq\bar{V}$ or until $d<d_{\max}$ ($d_{\max}$ presently set at 3).

6) If $d>d_{\max}$, set $n_3=n_3+1$ or $n_3=n_3+2$ depending upon whether the next mode retained by the model reduction criterion is real or complex, respectively. Return to step 2 with $d=0$. The nested iterations on d and n_3 continue until $V_{12}<\bar{V}$ or until $n_3=n_1$. If $n_3=n_1$, the performance requirement cannot be met with any reduced model.

This sequence of steps can be implemented automatically on a digital computer. The procedure attempts to retain the lowest number n_3 of physical modes for which model error compensation is possible with three or less synthetic modes from the model error system. There is no assurance that such a minimum is achieved, however. The algorithm has not optimized the sequence of iterations. It might be possible, for instance, to obtain better performance by iterating on n_3 first to get, in some application, $n_3=2$, $d=1$, instead of, say, $n_3=1$, $d=2$. Changing the sequence of iterations on n_3 and d presents an alternate solution for comparison. Figure 1 shows results obtained for a 14th-order flexible spacecraft system. The control system was unstable on the first iteration with $n_3=2$, $d=0$ (this corresponds to a Kalman filter based upon a rigid body model of the flexible spacecraft) but which was stable with the second iteration design $n_3=2$, $d=1$. That is, the model errors arising from the truncated modes were compensated with an adaptive Chebyshev filter of order 1. The Chebyshev-Fourier coefficient matrix P_z is updated every $\tau=2.5$ sec according to the formula (37) in Ref. 3, and after the third update of P_z the eigenvalues of the closed-loop system [Eq. (4)] have moved into the left half-plane. The spacecraft data are given by Eq. (38) in Ref. 3, with the single exception that $b=11.2$ in Eq. (38) to correspond to a high thrust condition.

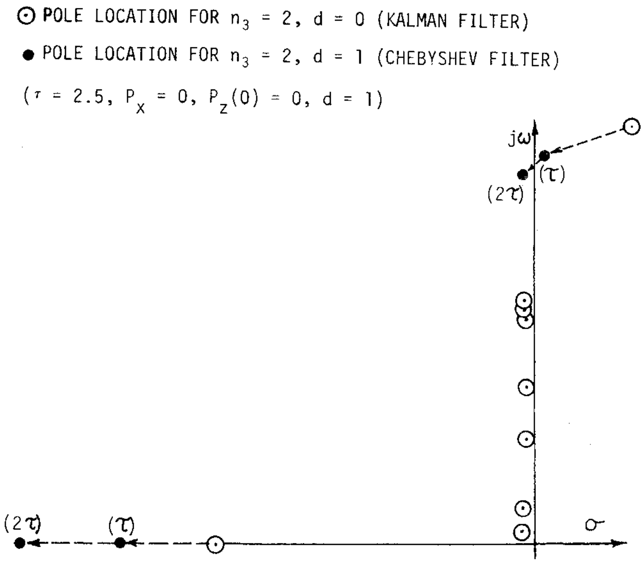

Fig. 1 Pole shift with adaptive Chebyshev filter.

Conclusions

A model reduction scheme based upon control performance sensitivity and a model error compensation scheme which curve fits the model error vector with orthogonal functions in real time have been combined in an ad hoc manner to form an algorithm for generating approximate solutions to the MCP with a performance constraint. While no theory yet vouches for the success of the combination of these two schemes, each scheme has theoretical support on its own merit as developed in Refs. 1 and 2. This represents an attempt to bring together the modeling and control problems in a framework that still can yield linear dynamical feedback time-invariant controllers, although an adaptive version is also possible.

Acknowledgment

C. Gregory has programmed the algorithm discussed herein and has provided the numerical results in this Note.

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Observer Stabilization of Singularly Perturbed Systems

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Introduction

IN a recent Note, Suzuki and Miura[1] solved the problem of stabilization of singularly perturbed linear systems by linear state feedback. They found that stabilization of the reduced system (fast modes eliminated) would guarantee the stability of the full system. A similar result was obtained by Porter.[2] Such results are useful in designing stable systems where fast and slow modes appear, e.g., systems with parasitic elements or control actuators. However, in most cases the full state vector is unavailable and must be reconstructed from available measurements by a Kalman filter[3] or a Luenberger observer.[4] In this Note, the feedback stabilization is developed for singularly perturbed systems where the state vector is reconstructed by a Luenberger observer based on the reduced system. It is shown, by the lemma of Klimushchev and Krasovskii,[5,10] that with this observer feedback the full system remains stable. To illustrate the use of the method, the results are specialized to systems with parasitic elements and to autopilot stabilization of systems with control actuators.

Main Result

The singularly perturbed systems under consideration have the following form

$$\dot{X}_1 = A_{11}X_1 + A_{12}X_2 + B_1 u \tag{1a}$$

$$\epsilon\dot{X}_2 = A_{21}X_1 + A_{22}X_2 + B_2 u \tag{1b}$$

$$y = C_1X_2 + C_2X_2 + Du \tag{1c}$$

where X_1 represents the state vector of the slow modes and X_2 represents that of the fast modes. The output vector y represents the available measured information and its entries are linear combinations of the internal states X_1 and X_2 and the control vector u. The parameter ϵ is nonnegative and represents the response time of the fast modes of the system. The set of Eq. (1) will be referred to as the *full system*.

Assume that A_{22} is stable (i.e., its spectrum is in the open left half-plane). The *reduced system* is obtained by ignoring the fast modes (i.e., setting $\epsilon = 0$) and this yields

$$\dot{\bar{X}}_1 = \bar{A}_{11}\bar{X}_1 + \bar{B}_1 u \tag{2a}$$

$$\bar{y} = \bar{C}_1\bar{X}_1 + \bar{D}u \tag{2b}$$

where

$$\bar{A}_{11} = A_{11} - A_{12}A_{22}^{-1}A_{21}$$

$$\bar{B}_1 = B_1 - A_{12}A_{22}^{-1}B_2$$

$$\bar{C}_1 = C_1 - C_2A_{22}^{-1}A_{21}$$

$$\bar{D} = D - C_2A_{22}^{-1}B_2$$

Suppose stabilization is obtained by state feedback from a Luenberger observer[4] designed for the reduced system (2) where the fast modes are ignored. This would be a practical approach for most deterministic systems where a feedback compensator was desired. The observer is a dynamic system:

$$\dot{\hat{X}}_1 = \bar{A}_{11}\hat{X}_1 + \bar{B}_1 u + K(y - \hat{y}) \tag{3a}$$

$$\hat{y} = \bar{C}_1\hat{X}_1 + \bar{D}u \tag{3b}$$

and the stabilizing control is

$$u = G\hat{X}_1 \tag{4}$$

This produces a feedback compensator of the form:

$$\dot{\hat{X}}_1 = [\bar{A}_{11} + \bar{B}_1G - K\bar{C}_1 - K\bar{D}G]\hat{X}_1 + Ky \tag{5a}$$

$$u = G\hat{X}_1 \tag{5b}$$

which accepts the system output y and produces the stabilizing control u.

Set $e_1 = \hat{X}_1 - X_1$ and use Eqs. (1, 3, and 4) to obtain

$$\dot{e}_1 = [\bar{A}_{11} - K\bar{C}_1 - (A_{12} - KC_2)A_{22}^{-1}B_2G]e_1$$
$$- (A_{12} - KC_2)A_{22}^{-1}(A_{21} + B_2G)X_1 - (A_{12} - KC_2)X_2 \tag{6}$$

Also use Eq. (4) in Eq. (1) to produce

$$\dot{X}_1 = (A_{11} + B_1G)X_1 + A_{12}X_2 + B_1Ge_1 \tag{7a}$$

$$\epsilon\dot{X}_1 = (A_{21} + B_2G)X_1 + A_{22}X_2 + B_2Ge_1 \tag{7b}$$

Let $Z = (X_1, e_1)^T$ and rewrite Eqs. (6) and (7) as

$$\dot{Z} = H_{11}Z + H_{12}X_2 \tag{8a}$$

$$\epsilon\dot{X}_2 = H_{21}Z + H_{22}X_2 \tag{8b}$$

where

$$H_{11} = \begin{bmatrix} A_{11} + B_1G & B_1G \\ -(A_{12} - KC_2)A_{22}^{-1}(A_{21} + B_2G) & T_{11} \end{bmatrix}$$

$$T_{11} = \bar{A}_{11} - K\bar{C}_1 - (A_{12} - KC_2)A_{22}^{-1}B_2G$$

Received June 27, 1977; revision received July 29, 1977.

Index categories: Guidance and Control; Spacecraft Dynamics and Control.
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